

Homework 6

Math 121A, Fall 2025.

Due: Wednesday, Oct 8, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (a) On this branch,

$$\log i = \log|i| + i \arg(i) = i\frac{\pi}{2}.$$

As a sanity check, $e^{\log i} = e^{i\pi/2} = i$.

(b) On this branch,

$$\log i = \log|i| + i(\arg(i) + 2\pi) = i\frac{5\pi}{2}.$$

As a sanity check, $e^{\log i} = e^{i5\pi/2} = e^{i\pi/2}e^{i2\pi} = i$.

(c) We have $i^i = e^{i \log i} = e^{i^2\pi/2} = e^{-\pi/2}$ for part (a), and $e^{ii5\pi/2} = e^{-5\pi/2}$ for part (b). Curiously, in any case we obtain a real number.

Problem 2. (You can easily plot these on Wolfram).

Problem 2.5.1: We can compute this by observing $1 + i = \sqrt{2}e^{i\pi/4}$, so

$$\frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\pi/4}.$$

(b) Problem 2.5.7: Let's write

$$\frac{3+i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6-3i+2i+1}{4+1} = \frac{7-i}{5}.$$

(c) Problem 2.5.34: Not sure why it encourages you to do this in your head...I wouldn't do that. Anyways, $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$, so

$$\left(\frac{1+i}{1-i}\right)^5 = \left(\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}}\right)^5 = (e^{i\pi/2})^5 = e^{i5\pi/2} = i.$$

(d) Problem 2.5.50: We are looking for solutions to the equation

$$|x+iy| = y - ix.$$

First, note that the right hand side is real, so this automatically forces $x = 0$. So,

$$|iy| = y,$$

or $|iy| = |y| = y$. Thus any nonnegative real number $y \geq 0$ satisfies this condition. So the only solutions are of the form $(0, y)$ with $y \geq 0$.

Problem 3. (a) Problem 2.14.21: Following the hint, we write $-1 = e^{i(\pi+k2\pi)}$, $k \in \mathbb{Z}$. Then the set of possible values are

$$\cos(i \log(e^{i(\pi+k2\pi)})) = \cos(i(i(\pi+k2\pi))) = \cos(-\pi - k2\pi) = \cos(-\pi) = -1,$$

where we have used that $\cos$ is 2π -periodic.

(b) Problem 2.14.24: Let's compare these examples, as they will provide the requisite examples. Part (b) is easier, so we start with that.

Part (b) i^{-1} is well-defined, it is just $-i$. But to compute i^i we must make a choice of branch of logarithm (twice, actually): let $\log_{(k)}$ denote the branch of logarithm given by $[k2\pi, (k+1)2\pi)$ where $k \in \mathbb{Z}$, so that the possible values are

$$i^i = e^{i \log_{(k)}(i)} = e^{i \cdot i(\pi/2+2\pi k)} = e^{-(\pi/2+2\pi k)}.$$

Then, to raise $(i^i)^i$, we must make another choice of branch. To make our point, it suffices to choose the principal branch $\log_{(0)}$ and the branch $\log_{(1)}$:

$$\begin{aligned} (i^i)^i &= (e^{-(\pi/2+2\pi k)})^i = e^{i \log_{(0)}(e^{-(\pi/2+2\pi k)})} = e^{i(3\pi/2)} = -i \\ (i^i)^i &= (e^{-(\pi/2+2\pi k)})^i = e^{i \log_{(1)}(e^{-(\pi/2+2\pi k)})} = e^{i(-\pi/2+2\pi i)} = -ie^{-2\pi}. \end{aligned}$$

For part (a), note that $(-i)^{(2+i)(2-i)} = (-i)^5$ is well-defined and has exactly one possible value, $-i$. The other expression again depends on two choices of branch of logarithm: first,

$$(-i)^{2+i} = e^{(2+i) \log_{(k)}(-i)} = e^{(2+i)(3\pi/2+2\pi k)} = e^{3\pi+2\pi k} e^{i(3\pi/2+2\pi k)} = e^{3\pi+2\pi k} e^{i3\pi/2} = -ie^{3\pi+2\pi k}.$$

We will be able to show that these two expressions are not equal by just choosing $k = -1$, where it is $-ie^\pi$ and $k = 0$ where it is $-ie^{3\pi}$. We now pick a branch yet again, picking the principal branch $\log_{(0)}$. Then

$$\begin{aligned} \text{For } k = -1: \quad [(-i)^{2+i}]^{2-i} &= e^{(2-i) \log_{(0)}(-ie^\pi)} = e^{(2-i)(\pi+i3\pi/2)} = e^{7\pi/2} \\ \text{For } k = 0: \quad [(-i)^{2+i}]^{2-i} &= e^{(2-i) \log_{(0)}(-ie^{3\pi})} = e^{(2-i)(3\pi+i3\pi/2)} = e^{15\pi/2}. \end{aligned}$$

Problem 4. These are just like from week 2.

(a) $\sum nz^n$ has radius of convergence $R = 1$ by the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)z^{n+1}}{nz^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)z}{n} \right| = |z|.$$

So, it converges if $|z| < 1$ and diverges if $|z| > 1$. On the boundary circle $|z| = 1$, it clearly diverges since e.g. if $z = 1$, then this is $\sum 1 = \infty$.

(b) $\sum \frac{z^n}{n^2}$ similarly has a radius of convergence of $R = 1$ by the ratio test. But on the circle $|z| = 1$, it converges since $\sum \frac{|z^n|}{n^2} = \sum \frac{1}{n^2}$ converges by p-series.

(c) $\sum \frac{z^n}{n}$ similarly has a radius of convergence of $R = 1$, but does not converge absolutely on the boundary circle since e.g. $z = 1$ corresponds to the harmonic series. (The other values of z are much more subtle, e.g. $z = -1$ converges conditionally by the alternating series test, but I only care about absolute convergence here.)

(d) $\sum n!z^n$ has a radius of convergence $R = 0$ by the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)!z^{n+1}}{n!z^n} \right| = \lim_{n \rightarrow \infty} n|z| = \infty.$$

At $|z| = 0$, it is exactly 0.

Problem 5. This is false. For instance, if $y \in \mathbb{R}$, then

$$|\sin(iy)| = \frac{1}{|2i|} |e^{i(iy)} - e^{-i(iy)}| = \frac{1}{2} |e^{-y} - e^y|$$

which is unbounded (it is hyperbolic sine).

Problem 6. $2e^{-i\pi/2} = 2(-i) = -2i$ in $x + iy$ form.

Problem 7. The solutions to $z^3 = i$ are given by De Moivre's formula. In polar form, $i = e^{i\pi/2}$. Using De Moivre's formula, the 3^{rd} roots of i are:

$$\exp\left(i\frac{\pi}{6} + \frac{4\pi k}{6}\right), \quad k = 0, 1, 2.$$

These are $e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$, and $e^{i9\pi/6} = -i$.

Problem 8. (a) Let $a_1, a_2 \in \mathbb{R}$ and let $z, w_1, w_2 \in \mathbb{C}$. Then, since the complex numbers form a field, we may write

$$\phi_z(a_1w_1 + a_2w_2) = z(a_1w_1 + a_2w_2) = a_1(zw_1) + a_2(zw_2) = a_1\phi_z(w_1) + a_2\phi_z(w_2),$$

which means that $\phi_z : \mathbb{C} \rightarrow \mathbb{C}$ is real linear.

(b) Let $w = a + ib$ be a complex number. In this basis, w has matrix form $[w] = \begin{bmatrix} a \\ b \end{bmatrix}$. Then, since $zw = (xa - yb) + i(ya + xb)$, the matrix for ϕ_z is exactly

$$[\phi_z] = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}. \quad (1)$$

Just to verify, we can check:

$$[zw] = [\phi_z][w] = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} xa - yb \\ ya + xb \end{bmatrix}.$$

(c) Let's compute, letting $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$, and we have

$$[\phi_{z_1}] \cdot [\phi_{z_2}] = \begin{bmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{bmatrix} = [\phi_{z_1z_2}], \quad (2)$$

which is what we wanted to show.

(d) $\det[\phi_z] = x^2 + y^2 = |z|^2$. This is exactly the absolute value squared of the complex number z .

(e) Assume that $|z| = 1$. This then implies that each column has norm 1, since $1 = |z|^2 = x^2 + y^2 = (-y)^2 + x^2$. Orthogonality then follows from seeing that the dot product of the columns is $x(-y) + yx = 0$.

Now assume that $[\phi_z]$ is an orthogonal matrix. Then we have that each column has norm 1, and so $x^2 + y^2 = 1$. This means that $|z| = 1$.

(f) We have from earlier $\sqrt{\det[\phi_z]} = |z|$, and so

$$\frac{1}{\sqrt{\det[\phi_z]}}[\phi_z] = \begin{bmatrix} \frac{x}{|z|} & \frac{-y}{|z|} \\ \frac{y}{|z|} & \frac{x}{|z|} \end{bmatrix}.$$

This is the matrix $[\phi_{z/|z|}]$, where $z/|z| = \frac{1}{|z|}(x + iy)$. Since $|z/|z|| = 1$, we have by part (e) that this matrix is orthogonal. Its determinant is 1, which means that it is a rotation matrix.

(g) This decomposition can be rewritten as

$$[\phi_z] = \sqrt{\det(z)} \cdot \frac{1}{\sqrt{\det(z)}}[\phi_z].$$

The first term is the absolute value $\sqrt{\det(z)} = |z| =: r$, and the second term is the matrix corresponding to a complex number on the unit circle $e^{i\theta}$. So this reflects the polar form of a complex number:

$$z = re^{i\theta}.$$

Problem 9. Let's think a bit about complex conjugation, $f(z) = \bar{z}$.

- (a) i. Complex conjugation acts as the identity map on $z \in L_R$, since $\bar{x} = x$. The line is unchanged.
- ii. Complex conjugation flips the imaginary line onto itself, since $\overline{iy} = -iy$.
- iii. Complex conjugation flips this line over the real axis, since $\overline{s(1+i)} = \bar{s}(\overline{1+i}) = \bar{s}(1-i)$.
- (b) This function is real linear, which can be readily seen by verifying that the matrix is given by

$$[f] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- (c) This matrix is orthogonal since its columns form an orthonormal basis. The determinant is $\det[f] = -1$.
- (d) This function is *not* complex linear, since e.g.

$$\overline{iz} = -iz,$$

which need not be equal to $i\bar{z}$.

- (e) Complex conjugation flips the complex plane over the real axis.
- (f) Observe that by real-linearity and that complex conjugation distributes over multiplication,

$$\overline{g(z)} = \sum_{n=0}^{\infty} \overline{a_n z^n} = \sum_{n=0}^{\infty} a_n \bar{z}^n = g(\bar{z}).$$

The exponential series is such an example, since $a_n = \frac{1}{n!}$ are real.

- (g) Consider the function which multiplies by i , which is certainly holomorphic. Then

$$\overline{h(z)} = \overline{iz} = -i\bar{z} \neq h(\bar{z}).$$